Membership problems in graph groups

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where $\equiv$ is the smallest congruence that contains all pairs

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Extreme cases:

- $\mathbb{G}(\Sigma, \emptyset) = F(\Sigma)$ (the free group generated by Σ)
- $\mathbb{G}(\Sigma, \Sigma \times \Sigma \setminus \mathrm{Id}_\Sigma) = \mathbb{Z}^{|\Sigma|}$ (the free abelian group of rank $|\Sigma|$)

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Rational subset membership problem for G:
INPUT: Word $w \in (\Sigma \cup \Sigma^{-1})^*$ and a finite automaton A over the alphabet $\Sigma \cup \Sigma^{-1}$
QUESTION: $h(w) \in h(L(A))$?

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Rational subset membership problem:
Decidable for free groups (Benois) and free abelian groups (folklore)

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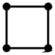

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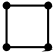

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For $L, K \subseteq \Sigma^*$ we have: $[L]_I \cap [K]_I \neq \emptyset \iff 1 \in h(LK^{-1})$.
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The class of transitive forests is the smallest class of graphs with:

1. A graph with one node is a transitive forest.
2. The disjoint union of two transitive forests is a transitive forest.
3. If G is a transitive forest and H results from G by adding a new node v and connecting v with all nodes from G, then also H is a transitive forest.

It suffices to show the following:

Let $\mathcal{G}$ be the smallest class of groups such that:

1. $\mathbb{Z} \in \mathcal{G}$
2. If $G_1, G_2 \in \mathcal{G}$ then also $G_1 * G_2 \in \mathcal{G}$ (closure under free products)
3. If $G \in \mathcal{G}$ then also $G \times \mathbb{Z} \in \mathcal{G}$

Then, for every group from $\mathcal{G}$ the rational subset membership problem is decidable.

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For $w \in \Theta^*$ and $\Delta \subseteq \Theta$ define the Parikh image $\Psi_\Delta(w) : \Delta \to \mathbb{N}$:

$$\Psi_\Delta(w)(a) = \text{number of occurrences of } a \text{ in } w$$

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1. If G is SLI, then G has a decidable rational subset membership problem.
2. If G is SLI, then $G \times \mathbb{Z}$ is SLI.
3. If G and H are SLI, then $G * H$ is SLI.

Submonoid membership problems

Submonoid membership problem for $\mathbb{G}(\Sigma, I)$:
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The rational subset membership problem for $\mathbb{G}(\mathrm{P4})$ can be reduced to the submonoid membership problem for $\mathbb{G}(\mathrm{P4})$.

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- Is decidability of the submonoid membership problem preserved by free products?
- If G has a decidable rational subset membership problem, is the same true for $G \times \mathbb{Z}$?