Star and Star Height Problems for Trace Monoids.

Daniel Kirsten

University Leipzig, Germany

October 11, 2008

Star and Star Height Problems for Trace Monoids.

Daniel Kirsten

University Leipzig, Germany

October 11, 2008

- D. Kirsten and G. Richomme. Decidability Equivalence between the Star Problem and the Finite Power Problem in Trace Monoids. *Theory of Computing Systems*, 34:3:193-227, 2001.
- D. Kirsten. The Star Problem and the Finite Power Property in Trace Monoids: Reductions Beyond C4. *Information and Computation*, 176:1:22-36, 2002.

Recognizability: Mezei/Wright 1967

$L \subseteq \mathbb{M}(A, D)$ is *recognizable*

$\iff$ $\exists$ *automaton* $[Q, h, F]$ with a finite monoid Q,
an epimorphism $h : \mathbb{M}(A, D) \to Q$,
$F \subseteq Q$, and $L = h^{-1}(F)$.

Recognizability: Mezei/Wright 1967

$L \subseteq \mathbb{M}(A, D)$ is *recognizable*

$\iff$ $\exists$ *automaton* $[Q, h, F]$ with a finite monoid Q, an epimorphism $h : \mathbb{M}(A, D) \to Q$, $F \subseteq Q$, and $L = h^{-1}(F)$.

or equivalently:

$\iff$ L is def. by a star-connected rational exp. (Ochmański 1984)

$\iff$ L is definable in MSOL. (Thomas 1990)

$\iff$ L is saturrated by a finite congruence.

$\iff$ the syntactic monoid of L is finite.

$\iff$ $[L]^{-1} \subseteq A^*$ is recognizable.

Recognizability: Mezei/Wright 1967

$L \subseteq \mathbb{M}(A, D)$ is *recognizable*

$$\iff \quad \exists \text{ automaton } [Q, h, F] \text{ with a finite monoid } Q,$$
$$\text{an epimorphism } h : \mathbb{M}(A, D) \to Q,$$
$$F \subseteq Q, \text{ and } L = h^{-1}(F).$$

Theorem 1: Mezei 1974

Let $\mathbb{M}(A, D) = \mathbb{M}(A_1, D_1) \times \mathbb{M}(A_2, D_2)$.

$$L \subseteq \mathbb{M}(A, D) \text{ is recognizable} \iff \bigcup_{\text{fin}} L_i \times R_i$$

for recognizable $L_i \subseteq \mathbb{M}(A_1, D_1)$

and recognizable $R_i \subseteq \mathbb{M}(A_2, D_2)$.

Recognizability: Mezei/Wright 1967

$L \subseteq \mathbb{M}(A, D)$ is *recognizable*

$\iff$ $\exists$ *automaton* $[Q, h, F]$ with a finite monoid Q, an epimorphism $h : \mathbb{M}(A, D) \to Q$, $F \subseteq Q$, and $L = h^{-1}(F)$.

Closure properties:

$\cup$, $\cap$, $\overline{}$, inverse homomorphisms, finite sets, concatenation.

Not closed under iteration and homomorphisms.

Example: $L = \{(a, b)\} \subseteq (a^* \times b^*)$

$$L^* = \{ (a^n, b^n) \mid n \in \mathbb{N} \}$$

$$[L^*]^{-1} = \{ w \in A^* \mid |w|_a = |w|_b \}$$

Star problem:

Ochmański 1984

Given:	recognizable trace language L
Question:	Is L^* recognizable?

Star problem: Ochmański 1984

Given: recognizable trace language L

Question: Is L^* recognizable?

Finite Power Problem: Brzozowski 1966

Given: recognizable trace language L

Question: Does some $n \in \mathbb{N}$ exist, such that $L^* = L^0 \cup L^1 \cup \cdots \cup L^n = L^{0,\dots,n}$?

...Finite Power Property (FPP)

Star problem: Ochmański 1984

Given:	recognizable trace language L
Question:	Is L^* recognizable?

Finite Power Problem: Brzozowski 1966

Given:	recognizable trace language L
Question:	Does some $n \in \mathbb{N}$ exist, such that $L^* = L^0 \cup L^1 \cup \cdots \cup L^n = L^{0,\ldots,n}$?
	... Finite Power Property (FPP)

Lemma 2:
Let $L \subseteq \mathbb{M}(A, D)$ be recognizable.
L has FPP $\implies$ L^* recognizable. □

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.

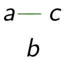
History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987: L^* is recognizable, if L is connected and recognizable.

History:

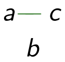
- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987:
  L^* is recognizable, if L is connected and recognizable.
- Ochmański 1990

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987:
  L^* is recognizable, if L is connected and recognizable.
- Ochmański 1990
- Sakarovitch 1992: The star problem is decid. in $\mathbb{M}(A, D)$ if 

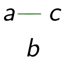
  does not occur in (A, D).

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987:
  L^* is recognizable, if L is connected and recognizable.
- Ochmański 1990
- Sakarovitch 1992: The star problem is decid. in $\mathbb{M}(A, D)$ if 

  does not occur in (A, D).
- Gastin/Ochmański/Petit/Rozoy 1992:
  The star problem is decidable in $A^* \times b^*$.

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987:
  L^* is recognizable, if L is connected and recognizable.
- Ochmański 1990
- Sakarovitch 1992: The star problem is decid. in $\mathbb{M}(A, D)$ if 

  does not occur in (A, D).
- Gastin/Ochmański/Petit/Rozoy 1992:
  The star problem is decidable in $A^* \times b^*$.
- Métivier/Richomme 1994:
  The FPP is decidable for connected, recognizable languages L.

History:

- Kleene 1956
- Ginsburg/Spanier 1966: Decidability in free commutative monoids.
- Simon 1978, Hashiguchi 1979: Decidab. of the FPP in free monoids.
- Ochmański 1984, Métivier 1986, Clerbout/Latteux 1987: L^* is recognizable, if L is connected and recognizable.
- Ochmański 1990
- Sakarovitch 1992: The star problem is decid. in $\mathbb{M}(A, D)$ if $a — c$ / b does not occur in (A, D).
- Gastin/Ochmański/Petit/Rozoy 1992: The star problem is decidable in $A^* \times b^*$.
- Métivier/Richomme 1994: The FPP is decidable for connected, recognizable languages L.
- Richomme 1994: Both problems are decidable in $\mathbb{M}(A, D)$ if $a — c$ / $b — d$ does not occur in (A, D). $\ldots \{a, c\}^* \times \{b, d\}^* =$ **C4**

Some results:

Theorem 3: Kirsten 1999

Let $\mathbb{M} = \mathbb{M}_1 \times \mathbb{M}_2$ be a trace monoid and $L \subseteq \mathbb{M}$ be a recognizable language with $L \subseteq (\mathbb{M}_1 \setminus 1) \times (\mathbb{M}_2 \setminus 1)$.

Then, L^* is recognizable **iff** L has the FPP. $\square$

Some results:

Theorem 3: Kirsten 1999

Let $\mathbb{M} = \mathbb{M}_1 \times \mathbb{M}_2$ be a trace monoid and $L \subseteq \mathbb{M}$ be a recognizable language with $L \subseteq (\mathbb{M}_1 \setminus 1) \times (\mathbb{M}_2 \setminus 1)$.

Then, L^* is recognizable **iff** L has the FPP. $\square$

Theorem 4: Kirsten/Richomme 2001

The trace monoids with a decidable star problem are exactly the trace monoids with a decidable FPP. $\square$

Some more results:

Theorem 5: Richomme 1994

If the star problem is decidable in $\mathbb{M}$, then it is decidable in $\mathbb{M} \times b^*$. □

Some more results:

Theorem 5: Richomme 1994

If the star problem is decidable in $\mathbb{M}$, then it is decidable in $\mathbb{M} \times b^*$. $\square$

Theorem 6: Richomme 1994

Let $\mathbb{M}(A, D)$ be a trace monoid.

The star problem is decidable in $\mathbb{M}(A, D)$ iff it is decidable for

- $L \subseteq \mathbb{M}(B, D)$ for every strict $B \subset A$.
- $L \subseteq \mathbb{M}(A, D)_{=A}$. $\square$

Remark:

$$\mathbb{M}(A, D)_{=A} \quad = \quad \mathbb{M}(A, D) \setminus \bigcup_{B \subset A} \mathbb{M}(B, D)$$

More recent results:

Let $\mathbb{K}_n = \{a_1, b_1\}^* \times \{a_2, b_2\}^* \times \cdots \times \{a_n, b_n\}^*$, i.e., $\mathbb{K}_2 \cong \mathrm{C4}$.

Theorem 7: Kirsten 2002

Let $n > 0$ and assume the decidability of the star problem in $\mathbb{K}_n$. Then, the star problem is decidable in any trace monoid without $\mathbb{K}_{n+1}$ submonoid. $\square$

Corollary:

Either the star problem is decidable in every trace monoid, **or** there is some $n > 1$ such that the trace monoids with an undecidable star problem are charactarized by a $\mathbb{K}_n$ submonoid. $\square$

Conclusions:

- Generalization of earlier results.

Open Problems:

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.

Open Problems:

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.

Open Problems:

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).

Open Problems:

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).
- Increasing concurrency is troublesome!

Open Problems:

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).
- Increasing concurrency is troublesome!

Open Problems:

- Decidability in $\mathrm{C4}$ is still open.

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).
- Increasing concurrency is troublesome!

Open Problems:

- Decidability in $\mathrm{C4}$ is still open.
- Reductions from $\mathbb{K}_{n+1}$ to $\mathbb{K}_n$.

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).
- Increasing concurrency is troublesome!

Open Problems:

- Decidability in $\mathrm{C4}$ is still open.
- Reductions from $\mathbb{K}_{n+1}$ to $\mathbb{K}_n$.
- Complexity issues.

Conclusions:

- Generalization of earlier results.
- Future research can restrict to $\mathbb{K}_n$.
- Importance of the FPP is shown again.
- Precise border (in comparison to the code problem).
- Increasing concurrency is troublesome!

Open Problems:

- Decidability in $\mathrm{C4}$ is still open.
- Reductions from $\mathbb{K}_{n+1}$ to $\mathbb{K}_n$.
- Complexity issues.
- Subclasses of recognizable languages (?)

Proposition 8:
Let (A, D) a dependence alphabet and let D be transitive.
Let n be the number of non-singleton components of (A, D).
If the FPP is decidable in $\mathbb{K}_n$,
then the FPP is decidable in $\mathbb{M}(A, D)$. □

Proof: ... obvious ...

Proposition 8:
Let (A, D) a dependence alphabet and let D be transitive.
Let n be the number of non-singleton components of (A, D).
If the FPP is decidable in $\mathbb{K}_n$,
then the FPP is decidable in $\mathbb{M}(A, D)$. □

Proof: ...obvious ...

Let $A_1, \ldots, A_n$ be the non-singleton components of (A, D),

and $b_1, \ldots, b_m$ be the remaining letters.

$\mathbb{M}(A, D) \cong A_1^* \times A_2^* \times \cdots \times A_n^* \times b_1^* \times \cdots \times b_m^*$,

Reduction to $A_1^* \times A_2^* \times \cdots \times A_n^*$ by Theorem 5,

injective homomorphism to $\mathbb{K}_n$. □

Proposition 9:
Let (A, D) a dependence alphabet.

If the FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$,

then it is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$. $\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{ (A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1} \right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

$\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{ (A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1} \right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

Induction: $|A| = 1$ ist obvious

$\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{ (A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1} \right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

Induction: $|A| = 1$ ist obvious

Let $(A, D) \in \mathcal{C}$ be arbitrary.

By induction, the FPP is decidable in $\mathbb{M}(B, D)$ for every $B \subset A$.

$\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{ (A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1} \right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

Induction: $|A| = 1$ ist obvious

Let $(A, D) \in \mathcal{C}$ be arbitrary.

By induction, the FPP is decidable in $\mathbb{M}(B, D)$ for every $B \subset A$.

By Thm. 6 it suffices to show the dec. of the FPP for $L \subseteq \mathbb{M}(A, D)_{=A}$.

$\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{(A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1}\right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

Induction: $|A| = 1$ ist obvious

Let $(A, D) \in \mathcal{C}$ be arbitrary.

By induction, the FPP is decidable in $\mathbb{M}(B, D)$ for every $B \subset A$.

By Thm. 6 it suffices to show the dec. of the FPP for $L \subseteq \mathbb{M}(A, D)_{=A}$.

By Prop. 9 $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ for $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$.

$\square$

Proof of Theorem 7: via FPP by Theorem 4...

Let $\mathcal{C} = \left\{ (A, D) \,\middle|\, \mathbb{M}(A, D) \text{ does not contain a } \mathbb{K}_{n+1} \right\}$.

$\mathcal{C}$ is subalphabet closed.

To show: decidability of the FPP in $\mathbb{M}(A, D)$ for every $(A, D) \in \mathcal{C}$

Induction: $|A| = 1$ ist obvious

Let $(A, D) \in \mathcal{C}$ be arbitrary.

By induction, the FPP is decidable in $\mathbb{M}(B, D)$ for every $B \subset A$.

By Thm. 6 it suffices to show the dec. of the FPP for $L \subseteq \mathbb{M}(A, D)_{=A}$.

By Prop. 9 ··· ··· ··· ··· ··· for $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$.

By Prop. 8 ··· ··· ··· ··· for $\mathbb{K}_k$, where k is the number of non-singleton components of $(A, \mathrm{tr}(D))$, i.e., of (A, D).

We have $k < n + 1$. □

Proposition 9:
Let (A, D) a dependence alphabet.
If the FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$,
then it is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$. □

Proposition 9:
Let (A, D) a dependence alphabet.
If the FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$,
then it is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$. □

Proof: ... inductive by Proposition 10 □

Proposition 9:
Let (A, D) a dependence alphabet.
If the FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, \mathrm{tr}(D))_{=A}$,
then it is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$. $\square$

Proof: ... inductive by Proposition 10 $\square$

Proposition 10:
Let (A, D) be a dependence alphabet.
Let $a, b, c \in A$ such that $a - b - c$ but $\neg\ a - c$.
Let $D' = D \cup \{a - c\}$.
If the FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, D')_{=A}$,
then it is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$. $\square$

Proof of Proposition 10:

FPP is decidable for recognizable $L \subseteq \left(\mathbb{M}(A, D')b\right)_{=A}$.

FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$.

Proof of Proposition 10:

FPP is decidable for recognizable $L \subseteq \left(\mathbb{M}(A, D')b\right)_{=A}$.

$\Downarrow$ Application of Proposition 12

FPP is decidable for recognizable $L \subseteq \left(\mathbb{M}(A, D)b\right)_{=A}$.

FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$.

Proof of Proposition 10:

FPP is decidable for recognizable $L \subseteq \big(\mathbb{M}(A, D')b\big)_{=A}$.

$\Downarrow$ Application of Proposition 12

FPP is decidable for recognizable $L \subseteq \big(\mathbb{M}(A, D)b\big)_{=A}$.

$\Downarrow$ Proposition 11

FPP is decidable for recognizable

$$L \subseteq \big(\mathbb{M}(A, D)\big)_{=A} b \big(\mathbb{M}(A, D)\big)_{=A}.$$

FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$.

Proof of Proposition 10:

FPP is decidable for recognizable $L \subseteq \left(\mathbb{M}(A, D')b\right)_{=A}$.

$\Downarrow$ Application of Proposition 12

FPP is decidable for recognizable $L \subseteq \left(\mathbb{M}(A, D)b\right)_{=A}$.

$\Downarrow$ Proposition 11

FPP is decidable for recognizable

$$L \subseteq \left(\mathbb{M}(A, D)\right)_{=A} b \left(\mathbb{M}(A, D)\right)_{=A}.$$

$\Downarrow$ L has FPP $\iff$ $(L^3 \cup L^4 \cup L^5)$ has FPP.

FPP is decidable for recognizable $L \subseteq \mathbb{M}(A, D)_{=A}$.

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\iff$ $\langle L \rangle^{-1}$ has FPP. $\quad \square$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\quad \Longleftrightarrow \quad \langle L\rangle^{-1}$ has FPP. $\quad \square$

Proof Idea:
$\ldots \Leftarrow \ldots$ $\; L = \langle\langle L\rangle^{-1}\rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L\rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$$(\langle L\rangle^{-1})^n \; = \quad (\langle L\rangle^{-1})^n$$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\quad \Longleftrightarrow \quad \langle L \rangle^{-1}$ has FPP. $\quad \square$

Proof Idea:
$\ldots \Leftarrow \ldots \;\; L = \langle\langle L \rangle^{-1}\rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L \rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$(\langle L \rangle^{-1})^n \; = \; \langle\langle(\langle L \rangle^{-1})^n\rangle\rangle^{-1}$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\iff$ $\langle L \rangle^{-1}$ has FPP. $\square$

Proof Idea:
$\ldots \Leftarrow \ldots$ $L = \langle \langle L \rangle^{-1} \rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L \rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$(\langle L \rangle^{-1})^n = \langle \langle (\langle L \rangle^{-1})^n \rangle \rangle^{-1} = \langle (\langle \langle L \rangle^{-1} \rangle)^n \rangle^{-1} = \langle L^n \rangle^{-1}$.

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\iff$ $\langle L \rangle^{-1}$ has FPP. $\quad \square$

Proof Idea:
$\ldots \Leftarrow \ldots$ $L = \langle \langle L \rangle^{-1} \rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L \rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$$(\langle L \rangle^{-1})^n = \langle \langle (\langle L \rangle^{-1})^n \rangle \rangle^{-1} = \langle (\langle \langle L \rangle^{-1} \rangle)^n \rangle^{-1} = \langle L^n \rangle^{-1}.$$

Hence, $(\langle L \rangle^{-1})^* = \langle L^* \rangle^{-1} =$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\iff$ $\langle L \rangle^{-1}$ has FPP. $\square$

Proof Idea:
$\ldots \Leftarrow \ldots$ $L = \langle \langle L \rangle^{-1} \rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L \rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$$(\langle L \rangle^{-1})^n = \langle \langle (\langle L \rangle^{-1})^n \rangle \rangle^{-1} = \langle (\langle \langle L \rangle^{-1} \rangle)^n \rangle^{-1} = \langle L^n \rangle^{-1}.$$

Hence, $(\langle L \rangle^{-1})^* = \langle L^* \rangle^{-1} = \langle L^{0,\ldots,n} \rangle^{-1} =$

Let $\langle\,\rangle : \mathbb{M}(A, D') \to \mathbb{M}(A, D)$ be the can. homomorphism.

$$[\,]_D \quad = \quad \langle\,\rangle \circ [\,]_{D'} \quad = \quad \langle [\,]_{D'} \rangle$$

Proposition 12:
Let $L \subseteq \mathbb{M}(A, D)b$: $\quad L$ has FPP $\iff$ $\langle L \rangle^{-1}$ has FPP. $\quad \square$

Proof Idea:
$\ldots \Leftarrow \ldots$ $\; L = \langle \langle L \rangle^{-1} \rangle$. Homomorphisms preserve the FPP.

$\ldots \Rightarrow \ldots$
For every $n \geq 0$, $(\langle L \rangle^{-1})^n$ is closed under commut. of a and c, i.e.,

$$(\langle L \rangle^{-1})^n = \langle \langle (\langle L \rangle^{-1})^n \rangle \rangle^{-1} = \langle (\langle \langle L \rangle^{-1} \rangle)^n \rangle^{-1} = \langle L^n \rangle^{-1}.$$

Hence, $(\langle L \rangle^{-1})^* = \langle L^* \rangle^{-1} = \langle L^{0,\ldots,n} \rangle^{-1} = (\langle L \rangle^{-1})^{0,\ldots,n}$.

Proof of Proposition 11:

Let $[Q, h, F]$ be an automaton such that $L = h^{-1}(F)$.

Let $g : \mathbb{M}(\Sigma, D) \to \mathbb{M}(\Sigma, D)$ defined by:

$$g(a) = a \qquad \text{for } a \neq b \text{ and}$$

$$g(b) = b^{|Q|+1}$$

Proof of Proposition 11:

Let $[Q, h, F]$ be an automaton such that $L = h^{-1}(F)$.

Let $g : \mathbb{M}(\Sigma, D) \to \mathbb{M}(\Sigma, D)$ defined by:

$$g(a) = a \qquad \text{for } a \neq b \text{ and}$$

$$g(b) = b^{|Q|+1}$$

g is injective and connected, thus $g(L)$ is recognizable.

$$L \text{ has FPP} \iff g(L) \text{ has FPP}$$

We consider $g(L)$.

Proof of Proposition 11:

Let $[Q, h, F]$ be an automaton such that $L = h^{-1}(F)$.

Let $g : \mathbb{M}(\Sigma, D) \to \mathbb{M}(\Sigma, D)$ defined by:

$$g(a) = a \qquad \text{for } a \neq b \text{ and}$$

$$g(b) = b^{|Q|+1}$$

g is injective and connected, thus $g(L)$ is recognizable.

$$L \text{ has FPP} \iff g(L) \text{ has FPP}$$

We consider $g(L)$.

Let $\# : Q \to \{1, \ldots, |Q|\}$ an enumeration of Q, denoted by $\#q$.

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

,

□

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\; mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T \;=\; b\,\mathbb{M}\,a\Big(A \setminus b\Big)^*\Big\{b^{|Q|+1}\Big\}^* b^{1,\ldots,|Q|}\Big(A \setminus b\Big)^* a\,\mathbb{M}\,b$$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\# q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\# m} \cdot g(h^{-1}(p))$$

$$T = b\,\mathbb{M}\,a\Big(A \setminus b\Big)^* \Big\{b^{|Q|+1}\Big\}^* b^{1,\dots,|Q|}\Big(A \setminus b\Big)^* a\,\mathbb{M}\,b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\# q}$$

$$L_2 = \bigcup_{m, p \in Q,\; mh(b)p \in F} b^{|Q|+1-\# m} \cdot g(h^{-1}(p))$$

$$T = b\,\mathbb{M}\,a\Big(A \setminus b\Big)^* \Big\{b^{|Q|+1}\Big\}^* b^{1,\dots,|Q|}\Big(A \setminus b\Big)^* a\,\mathbb{M}\,b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

$g(L)^* \subseteq (L_1 L_2)^*$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T = b\,\mathbb{M}\,a\Big(A \setminus b\Big)^*\Big\{b^{|Q|+1}\Big\}^* b^{1,\dots,|Q|}\Big(A \setminus b\Big)^* a\,\mathbb{M}\,b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^* L_2$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\ mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T = b \mathbb{M} a \left(A \setminus b\right)^* \left\{b^{|Q|+1}\right\}^* b^{1,\ldots,|Q|} \left(A \setminus b\right)^* a \mathbb{M} b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\ldots \Leftarrow \ldots$

$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^* L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1 \cup T)^{0,\ldots,n} L_2$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T = b \, \mathbb{M} \, a \Big(A \setminus b \Big)^* \Big\{ b^{|Q|+1} \Big\}^* b^{1,\dots,|Q|} \Big(A \setminus b \Big)^* a \, \mathbb{M} \, b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

$$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^* L_2$$

$$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1 \cup T)^{0,\dots,n} L_2$$

$$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^{0,\dots,n} L_2 \cup T$$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T = b\,\mathbb{M}\,a\Big(A \setminus b\Big)^*\Big\{b^{|Q|+1}\Big\}^* b^{1,\ldots,|Q|}\Big(A \setminus b\Big)^* a\,\mathbb{M}\,b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\ldots \Leftarrow \ldots$

$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1(L_2 L_1)^* L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1(L_2 L_1 \cup T)^{0,\ldots,n} L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1(L_2 L_1)^{0,\ldots,n} L_2 \cup T \subseteq (L_1 L_2)^{0,\ldots,n+1} \cup T$

,

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\# q}$$

$$L_2 = \bigcup_{m,p \in Q,\ mh(b)p \in F} b^{|Q|+1-\# m} \cdot g(h^{-1}(p))$$

$$T = b \mathbb{M} a \Big(A \setminus b\Big)^* \Big\{b^{|Q|+1}\Big\}^* b^{1,\dots,|Q|} \Big(A \setminus b\Big)^* a \mathbb{M} b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^* L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1 \cup T)^{0,\dots,n} L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^{0,\dots,n} L_2 \cup T \subseteq (L_1 L_2)^{0,\dots,n+1} \cup T$

$\subseteq g(L)^{0,\dots,n+1} \cup T,$

$\square$

$$L_1 = \bigcup_{q \in Q} g(h^{-1}(q)) \cdot b^{\#q}$$

$$L_2 = \bigcup_{m,p \in Q,\, mh(b)p \in F} b^{|Q|+1-\#m} \cdot g(h^{-1}(p))$$

$$T = b \mathbb{M} a \Big(A \setminus b\Big)^* \Big\{b^{|Q|+1}\Big\}^* b^{1,\dots,|Q|} \Big(A \setminus b\Big)^* a \mathbb{M} b$$

One can show: $g(L)$ has FPP $\iff$ $L_2 L_1 \cup T$ has FPP.

$\dots \Leftarrow \dots$

$g(L)^* \subseteq (L_1 L_2)^* \subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^* L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1 \cup T)^{0,\dots,n} L_2$

$\subseteq \varepsilon \cup (L_1 L_2) \cup L_1 (L_2 L_1)^{0,\dots,n} L_2 \cup T \subseteq (L_1 L_2)^{0,\dots,n+1} \cup T$

$\subseteq g(L)^{0,\dots,n+1} \cup T$,

i.e., $g(L)^* = g(L)^{0,\dots,n+1}$. $\square$