A New Algebraic Framework for Recognizable Trace Languages

Manfred Kufleitner
Universität Stuttgart

joint work with Pascal Weil

DNTTT'08 Cremona, October 9, 2008

Introduction

- Algebraic approach: very powerful for regular word languages.
- A lot of universal tools have been developed.
- For recogizable trace languages, most of these tools do not work.
- Algebraic methods have been applied in the study of recognizable trace languages, but on a more combinatorial level.
- Ordinary monoids seem to be an insufficient model.
- Our approach: equip monoids with some notion of independence: **independence monoids**.

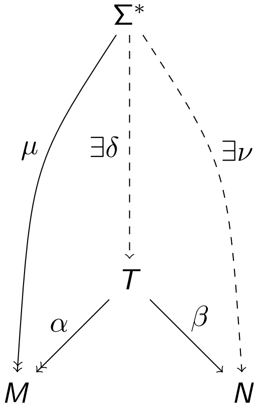
First application of our techniques

Theorem

If $\mathbb{B}(\Sigma_n[E])$ *is a variety that corresponds to* $\mathbf{B\Sigma_n}$ *then* $\Sigma_{n+1}[E] = \mathrm{Pol}\,\mathbb{B}(\Sigma_n[E])$ *and this positive variety corresponds to the variety of ordered l-monoids* $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!] \, \textcircled{M} \, \mathbf{B\Sigma_n}$.

Example

- Simple construction for Mal'cev products and word languages:



- $\mu^{-1}\nu = (\delta\alpha)^{-1}\delta\beta = \alpha^{-1}\delta^{-1}\delta\beta = \alpha^{-1}\beta$

How to equip monoids with an independence relation

- We already know two sorts of independence monoids:
  - trace monoids $\mathbb{M}(\Sigma, I)$
  - alphabet information of trace monoids: $(2^{\Sigma}, \cup, J)$ where $(A, B) \in J$ if $A \times B \subseteq I$
- Independence implies commutativity – but not vice versa.
- Special role of the empty trace.
- What else?

Axioms of I-monoids (M, I)

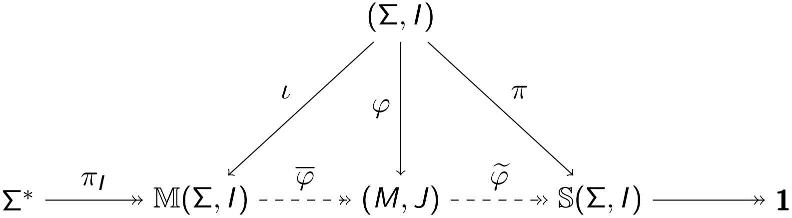
- M monoid
- $I \subseteq M \times M$ symmetric
- $(a, b) \in I$ implies $ab = ba$ in M
- $(a, 1) \in I$
- $(a, a) \in I$ if and only if $a = 1$
- $(a, bc) \in I$ if and only if $(a, b) \in I$ and $(a, c) \in I$

Products, homomorphisms, ...

- I-submonoids are straightforward.
- Products: n-tuples are independent if all components are independent.
- Homomorphisms $\varphi : (M, I) \to (N, J)$ come in two flavors:
  - *weak*: $(a, b) \in I$ implies $\big(\varphi(a), \varphi(b)\big) \in J$
  - *strong*: $(a, b) \in I$ if and only if $\big(\varphi(a), \varphi(b)\big) \in J$
- Syntactic I-congruence of $L \subseteq (M, I)$:
  $a \sim_L b$ if for all $x, y \in M$:
  - $xay \in L$ if and only if $xby \in L$
  - $(x, a) \in I$ if and only if $(x, b) \in I$

Universal properties

- An independence alphabet (Σ, I) strongly generates an I-monoid (M, J) if there is a mapping $\varphi : \Sigma \to M$ such that
  - $\varphi(\Sigma)$ generates M.
  - $(a, b) \in I$ if and only if $\big(\varphi(a), \varphi(b)\big) \in J$.



No Eilenberg Theorem for monoids

- Since every free monoid is a trace monoid, $\mathcal{V} \mapsto \mathbf{V}$ is surjective (i.e., $\mathbf{V}$ is the variety of monoids generated by the syntactic monoids of languages in $\mathcal{V}$)
- **But:** $\mathcal{V} \mapsto \mathbf{V}$ is not injective.
- $\mathbf{V} \mapsto \mathcal{V}$ is injective (i.e., $\mathcal{V}$ contains all languages which are recognizable by some monoid in $\mathbf{V}$), but not surjective.

I-varieties

- Language level:
  - Boolean operations.
  - Left- and right-quotients.
  - Inverse **weak** I-homomorphisms.
- Finite I-monoid level:
  - Submonoids.
  - Finite Products.
  - **Strong** I-homomorphic images.
- Recognition level:
  - (M, J) recognizes $L \subseteq \mathbb{M}(\Sigma, I)$ if there exists a **weak** I-homomorphism $\varphi : \mathbb{M}(\Sigma, I) \to (M, J)$ such that $\varphi^{-1}\varphi(L) = L$.
  - From weak to strong homomorphisms: $\widehat{\varphi} : \mathbb{M}(\Sigma, I) \to (M, J) \times \mathbb{S}(\Sigma, I)$ is **strong**.
  - Syntactic I-homomorphism is **strong**.

Eilenberg Theorem for I-monoids

Theorem
The function $\mathbf{V} \to \mathcal{V}$ *defines a bijection between the I-varieties that contain all finite skeleton monoids and the trace language varieties that contain* $\mathcal{S}$*.*

Theorem
The function $\mathbf{V} \to \mathcal{V}$ *defines a bijection between the* **ordered** *I-varieties that contain all finite skeleton monoids and the* **positive** *trace language varieties that contain* $\mathcal{S}$*.*

Equational descriptions of varieties

- Equations: $[u = v]_{(\Sigma,I)}$ where $u, v \in \mathbb{M}(\Sigma, I)$.
- (M, J) satisfies $[u = v]_{(\Sigma,I)}$ if for every weak I-homomorphism $\varphi : \mathbb{M}(\Sigma, I) \to (M, J)$ we have $\varphi(u) = \varphi(v)$.
- $[\![\, u = v \,]\!]_{(\Sigma,J)} \;=\; \{ (M, I) \mid (M, I) \text{ satisfies } [u = v]_{(\Sigma,J)} \}$

Theorem

If a class of I-monoids **V** *is defined ultimately by a sequence of equations then* **V** *is an I-variety.*

Theorem

Every I-variety that contains all finite skeleton monoids is ultimately defined by a sequence of equations.

- Example: $[\![\, zxzyz = zyzxz \,]\!]$ is equal to $[\![\, xy = yx \,]\!]$.
- $[\![\, zxzyz = zyzxz \,]\!]_{(x,y)\in I}$ is a different I-variety.
- Ordered versions do also exist.

Polynomial Closure

- $\mathrm{Pol}\mathcal{V}$: finite unions of languages of the form

$$L_0 a_1 L_1 \cdots a_k L_k$$

where $a_i \in \Sigma$ and $L_i \in \mathcal{V}$.

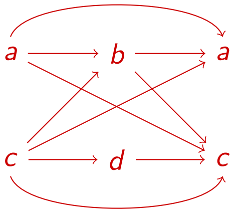
Theorem

Let $\mathbf{V}$ *be an I-variety which contains all skeleton monoids and let* $\mathbf{V} \rightarrow \mathcal{V}$. *Then* $[\![x^\omega y x^\omega \leq x^\omega]\!] \textcircled{M} \mathcal{V}$ *corresponds to* $\mathrm{Pol}\mathcal{V}$.

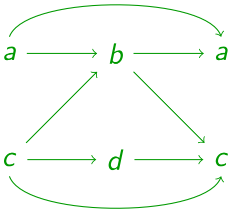
Example

Let $D = a—b—c—d$ and $t = acdbca$.

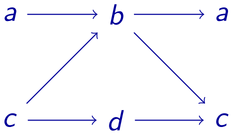
partial order:



dependence graph:



Hasse diagram:



First-order alternation hierarchy over dependence graphs

- First-order fragment $\Sigma_n[E]$ over dependence graphs: $L = L(\varphi)$ for some FO-formula φ in prenex normal form, with n blocks of quantifiers, starting with an existential block.

Theorem

If $\mathbb{B}(\Sigma_n[E])$ is a variety that corresponds to $\mathbf{B\Sigma_n}$ then $\Sigma_{n+1}[E] = \mathrm{Pol}\,\mathbb{B}(\Sigma_n[E])$ and this positive variety corresponds to the variety of ordered I-monoids $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!] \circledM \mathbf{B\Sigma_n}$.

Work in progress

- Wreath product principle for I-monoids (joint work with Bharat Adsul).
- Schützenberger product for I-monoids.

Future work (and far goals)

- Delay-Theorem.
- First-order fragments over Hasse-diagrams and partial orders.
- Krohn-Rhodes decomposition (at least for aperiodic monoids).
- Reiterman-Theorem.

Thank you!