When does partial commutative closure preserve regularity?

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Summary

(1) Algebraic tools
(2) Two questions, known results
(3) Commutative closure
(4) Closure under partial commutation
(5) Summary and open problems

Partial commutation

Let A be a finite alphabet and let I be an independence relation. We denote by $\sim_I$ the congruence on A^* generated by the set

$$\{ab = ba \mid (a, b) \in I\}$$

If L is a language on A^*, we denote by $[L]_I$ the closure of L under $\sim_I$ (I-commutation). If I is the total commutation, we simplify the notation to $\sim$ and $[L]$.

The dependence relation is $D = (A \times A) \setminus I$.

Syntactic ordered monoid

- The syntactic preorder of L is the relation $\leqslant_L$ defined by : $u \leqslant_L v$ iff, for every $x, y \in A^*$,

$$xvy \in L \Rightarrow xuy \in L$$

- The syntactic congruence of L is the relation $\sim_L$ defined by : $u \sim_L v$ iff, for every $x, y \in A^*$,

$$xvy \in L \Leftrightarrow xuy \in L$$

- The syntactic ordered monoid of L is $(A^*/{\sim_L}, \leqslant_L /{\sim_L})$

An equivalent definition

If $\mathcal{A}$ is a minimal deterministic automaton, the relation $\leqslant$ defined on Q by $p \leqslant q$ iff

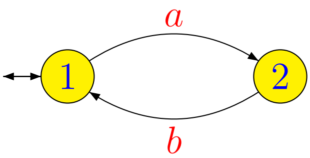
for all $u \in A^*$, $q{\cdot}u \in F \Rightarrow p{\cdot}u \in F$

is the syntactic order.

The syntactic ordered monoid of a language is the transition monoid of its ordered minimal automaton. The order is defined by

$$u \leqslant v \text{ iff, for all } q \in Q,\ q{\cdot}u \leqslant q{\cdot}v$$

The syntactic ordered monoid of $(ab)^*$



Order on the states :
$1 < 0$ and $2 < 0$

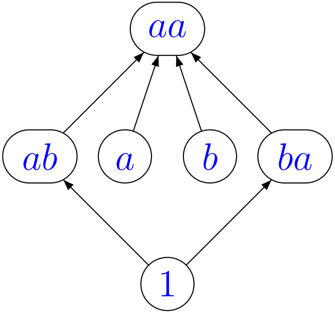
Elements

	1	2
1	1	2
a	2	0
b	0	1
aa	0	0
ab	1	0
ba	0	2

Relations

$bb = aa$
$aba = a$
$bab = b$

Order

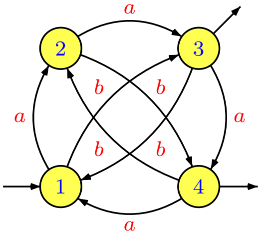


Group languages

A group language is a regular language whose syntactic monoid is a group, or, equivalently, is recognized by a deterministic automaton in which each letter defines a permutation of the set of states.

Examples on the alphabet $A = \{a, b\}$.

- {the words of odd length}
- $\{u \mid |u|_a + 2|u|_b \equiv 3 \bmod 5\}$
- $\{u \mid \binom{u}{ab} \equiv 1 \bmod 8\}$
- $(a(ab + bb)^*Aa)^*$



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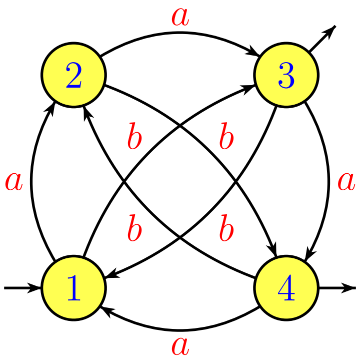
Use GasTeX !



Exponent and period of a regular language L

The exponent of L is the least integer ω such that, for all $u \in A^*$, $u^\omega \sim_L u^{2\omega}$.

The period of L is the least integer p for which, for all $u \in A^*$, $u^\omega \sim_L u^{\omega+p}$.

A language has period 1 iff it is star-fre (Schützenberger 65).

The period of a group language recognized by a group G divides $|G|$.

Two problems

- When is the closure of a regular language under [partial] commutation still regular?

- Are there any robust classes of languages closed under [partial] commutation?

Robust class = closed under some of the usual operations on languages, such as Boolean operations, product, star, shuffle, morphisms, inverses of morphisms, quotients, etc.

Known results on the first problem

Theorem (Ginsburg-Spanier 66, Gohon 85)

One can effectively decide whether the commutative closure of a regular language is regular or not.

Theorem (Sakarovitch 1992)

One can decide whether the closure $[L]_I$ of a regular language L is regular iff I is a transitive relation.

Star-free languages

Theorem (Muscholl, Petersen 1996)

One can decide whether the closure $[L]_I$ of a star-free language L is star-free iff I is a transitive relation.

Theorem (Muscholl, Petersen 1996)

Let L be a star-free language. If D is transitive, then $[L]_I$ is either star-free or non regular. If D is not transitive, then there exist star-free languages such that $[L]_I$ is regular but not star-free.

A convenient definition

Let $\mathcal{L}$ be a class of regular languages. We denote by $\mathrm{Pol}(\mathcal{L})$ the polynomial closure of $\mathcal{L}$, consisting of the finite unions of languages of the form

$$L_0 a_1 L_1 \cdots a_k L_k$$

where $a_1, \ldots, a_k \in A$ and $L_0, \ldots, L_k \in \mathcal{L}$.

Five classes of languages related to Problem 2

(1) $\mathsf{Pol}(\mathcal{I})$ = finite unions of languages of the form $A^*a_1A^* \cdots a_kA^*$,

(2) $\mathcal{J}$ = Boolean closure of $\mathsf{Pol}(\mathcal{I})$ (piecewise testable languages),

(3) $\mathsf{Pol}(\mathcal{J})$ = finite unions of languages of the form $A_0^*a_1A_1^* \cdots a_kA_k^*$ with $A_i \subseteq A$. Also called APC (Alphabetic Pattern Constraints) by Bouajjani, Muscholl and Touili.

(4) $\mathsf{Pol}(\mathcal{C}om)$ = polynomials of commutative languages.

Polynomial of group languages

Theorem (Pin, ICALP 1994)

Let L be a regular language. The following conditions are equivalent:

(1) *L is a polynomial of group languages,*

(2) *the ordered syntactic monoid of L satisfies the identity $x^\omega \leqslant 1$,*

(3) *L is open in the profinite topology.*

Known results on the second problem

We now summarise the results of Guaiana, Restivo and Salemi (00/04), Bouajjani, Muscholl and Touili (01/07) and Cécé, Héam and Mainier (07).

Theorem (Four classes Theorem)

Let I be any independence relation. Then

(1) *Pol*$(\mathcal{I})$ *is closed under commutation,*

(2) $\mathcal{J}$ *is closed under commutation,*

(3) *Pol*$(\mathcal{J})$ *is closed under I-commutation,*

(4) *Pol*$(\mathcal{C}om)$ *is closed under I-commutation.*

Our first result

Theorem

The commutative closure of a group language is regular.

Main argument: Insertion systems of Bucher, Ehrenfeucht and Haussler [1985].

Let π be a monoid morphism from A^* onto a finite group G and let $R = \pi^{-1}(1)$. Let us write $u \longrightarrow_R v$ if $u = u'u''$ and $v = u'ru''$ for some $r \in R$.

Let $\xrightarrow{*}_R$ be the reflexive transitive closure of the relation $\longrightarrow_R$. Then $\xrightarrow{*}_R$ is a well preorder.

A double improvement

Theorem

The commutative closure of a polynomial of group languages is also a polynomial of group languages.

Main argument: A refined description of the polynomials of group languages [Pin 1996] as a finite union of languages of the form $Ra_1R \cdots Ra_nR$, where R is a group language.
Then show that $[L]$ is closed under $\xrightarrow{*}_R$.

The positive variety $\mathcal{W}$

A positive variety of languages is a class of regular languages closed under union, intersection, quotients ($L \to u^{-1}L, Lu^{-1}$) and inverses of morphisms.

Theorem (Cano Gómez-Pin, 2002)

There exists a unique maximal positive variety $\mathcal{W}$ which does not contain the language $(ab)^*$, for all letters $a \neq b$.

Further, $\mathcal{W}$ is decidable, but this requires an algebraic characterization.

Algebraic characterization of $\mathcal{W}$

By a general result, a language belongs to $\mathcal{W}$ iff its syntactic ordered monoid belongs to a certain variety of finite ordered monoids **W**.

Theorem (Cano Gómez-Pin, 2002)

A finite ordered monoid M belongs to **W** *iff, for all pair (a, b) of elements of M such that $aba = a$ and $bab = b$, for all z of the minimal ideal of the submonoid of M generated by a and b, one has $(abzab)^\omega \leqslant ab$.*

The positive variety $\mathcal{W}$ is very robust!

- $\mathcal{W}$ is the unique maximal proper variety closed under shuffle,
- $\mathcal{W}$ is closed under product,
- $\mathcal{W}$ is closed under length decreasing morphism,
- $\mathcal{W}$ contains the group languages and their polynomial closure.
- $\mathcal{W}$ contains our "Four classes" and the languages whose syntactic monoid belongs to the variety **DS**.

Example. By definition, $(ab)^* \notin \mathcal{W}$. However $(ab)^* + A^*aaA^* \in \mathcal{W}$.

Main result

Theorem

If L is a language of $\mathcal{W}$, then $[L]$ is also in $\mathcal{W}$ (and hence is regular) and its period divides that of L.

Proof. Relies on the algebraic characterisation, which makes it difficult.

Corollary

The class of star-free languages of $\mathcal{W}$ is closed under commutation.

The case where I is transitive

Theorem

Let I be a transitive relation. If L is a language of $\mathcal{W}$, then $[L]_I$ is regular.

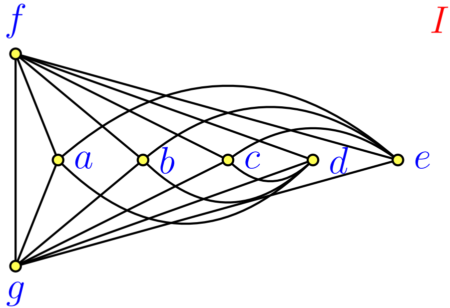
However, we don't know whether $[L]_I$ also belongs to $\mathcal{W}$.

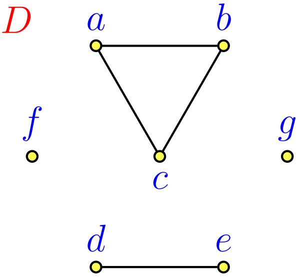
Proof. Makes use of ordered generalised automata (transitions are labeled by regular languages).

The case where D is transitive

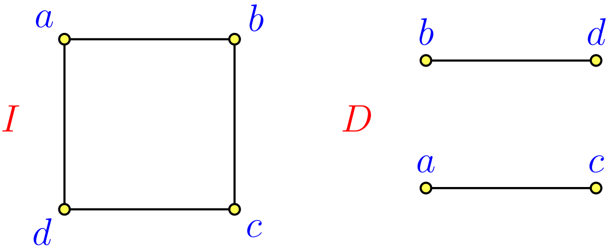
Theorem

Suppose that D is transitive. If L is a polynomial of group languages, then $[L]_I$ is also a polynomial of group languages.





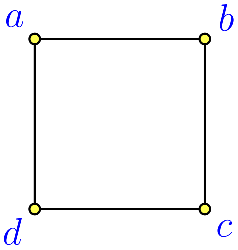
This result does not extend to $\mathcal{W}$.



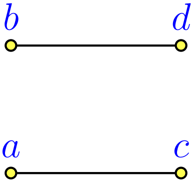
Here D is transitive. The language
$L = (abcd)^* + A^*aaA^* + A^*bbA^* + A^*ccA^* + A^*ddA^*$
$+ A^*ababA^* + A^*bcbcA^* + A^*cdcdA^* + A^*dadaA^*$
is in $\mathcal{W}$, but $[L]_I$ is not regular.

This result does not extend to $\mathcal{W}$.

I



D



Here D is transitive. The language
$L = (abcd)^* + A^*aaA^* + A^*bbA^* + A^*ccA^* + A^*ddA^*$
$+ A^*ababA^* + A^*bcbcA^* + A^*cdcdA^* + A^*dadaA^*$
is in $\mathcal{W}$, but $[L]_I$ is not regular.
It is not so easy to show that L is in $\mathcal{W}$, since its syntactic monoid has 170 elements...

Another extension

Let (A_1, I_1), ..., (A_k, I_k) be the connected components of the graph (A, I) and let (A_j, D_j) be the complement graph of (A_j, I_j).

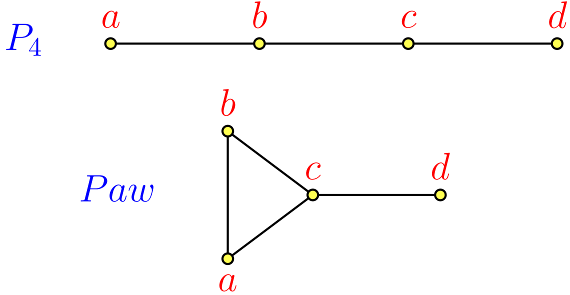
Theorem

If each graph (A_j, D_j) is transitive and L is a polynomial of group languages, then $[L]_I$ is regular.

A graph theoretic characterization

Theorem

The graphs (A_j, D_j) *are all* *transitive* *iff the graph* (A, I) *is* (P_4, Paw)*-free.*



Summary of our results

- Total commutation :
  if L is in $\mathcal{W}$, then so is $[L]$.
- I transitive :
  if L is in $\mathcal{W}$, then $[L]_I$ is regular.
- D transitive :
  if L is a polynomial of group languages, then so is $[L]_I$.
- (A, I) is (P_4, Paw)-free :
  if L is a polynomial of group languages, then $[L]_I$ is regular.

Conclusion and open problems

- The robust class $\mathcal{W}$ might offer an appropriate framework for modeling certain problems arising in the verification of concurrent systems.

- Open problems
  - If I is transitive and if L is in $\mathcal{W}$, does $[L]_I$ belong to $\mathcal{W}$?
  - If L is a group language, is $[L]_I$ always regular?
  - Is $\mathrm{Pol}(\mathcal{G})$ closed under partial commutations?
  - What about the polynomial closure of the family commutative languages $+$ group languages?