Approximation of the mean speedup in trace monoids and random generation of traces

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Developments and New Tracks in Trace Theory

ESF AutoMathA project 8 - 10, Cremona (Italy)

Brief history

Sequential Processes $\Longrightarrow$ Free Monoids

$\Sigma = \{a, b, c, d, e\}$
$w = acabde \in \Sigma^*$

Concurrent Processes $\Longrightarrow$ Trace Monoids

$(\Sigma, C) =$

[Mazurkiewicz, 77]

$t = [acabde] \in \Sigma^* / \equiv,$
$\equiv \leftarrow \{xy = yx \mid xCy\}$

Originally, they were introduced in [Cartier/Foata, 69]

Combinatorial Context $\Longrightarrow$ Part. Comm. Monoids

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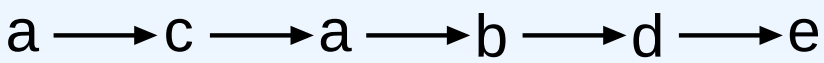

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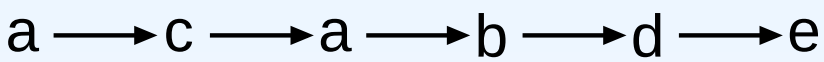

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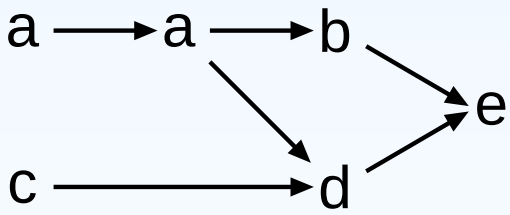

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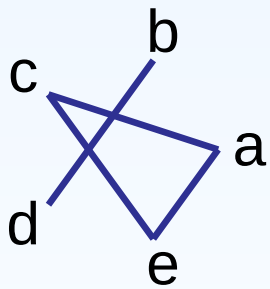

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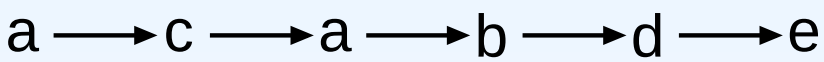

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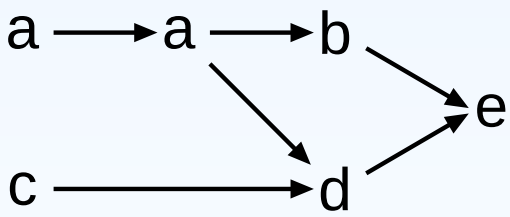

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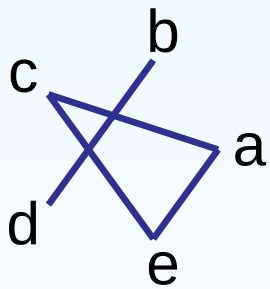

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A trace t is an equivalence class in Σ^*, $t = [w]$ where $w \in \Sigma^*$.

$$[acabde] = \{aacbde, aacdbe, acabde, acadbe, caabde, caadbe\}$$

The Foata Normal Form of a trace t is a univocal representation of t by means of cliques in (Σ, C):

- A pair of cliques (c, c') is *admissible* ($\mathrm{adm}(c, c')$) if $\forall \sigma' \in c', \exists \sigma \in c \mid (\sigma, \sigma') \notin C$.
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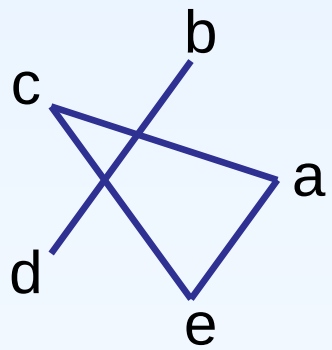

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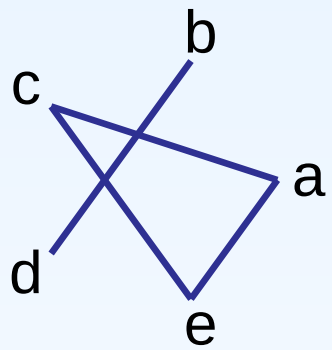

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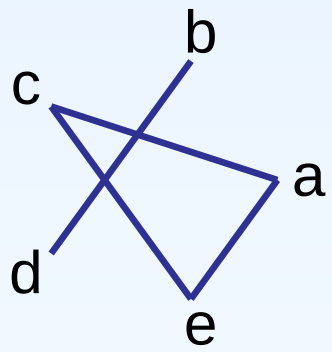

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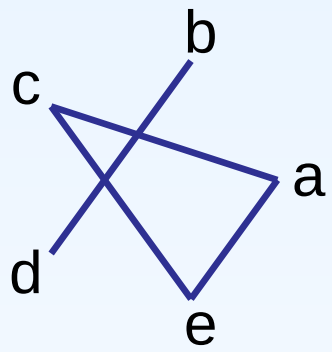

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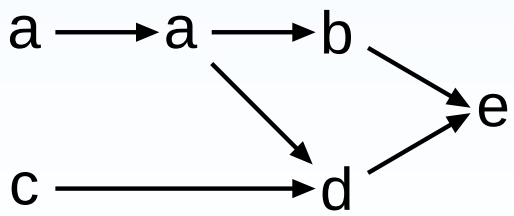

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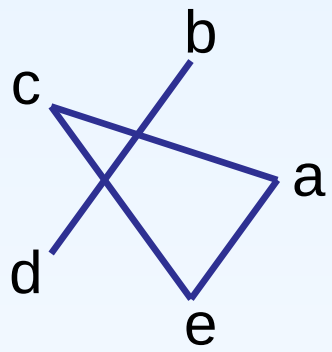

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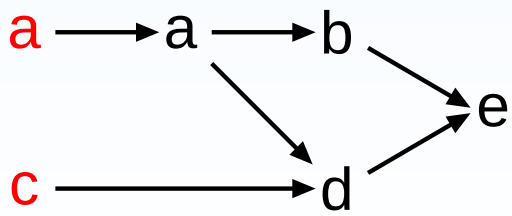

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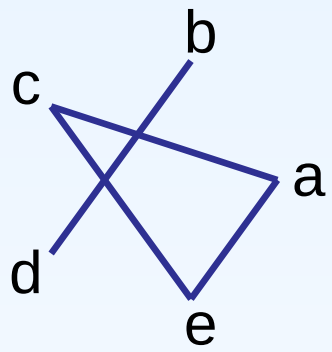

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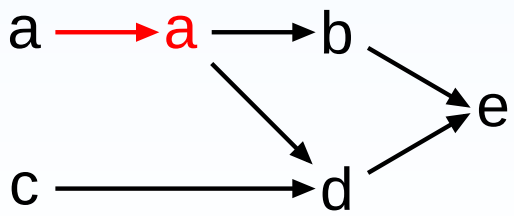

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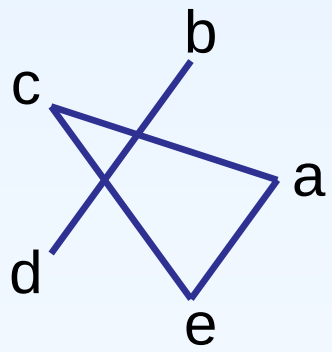

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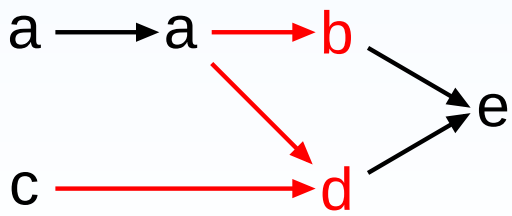

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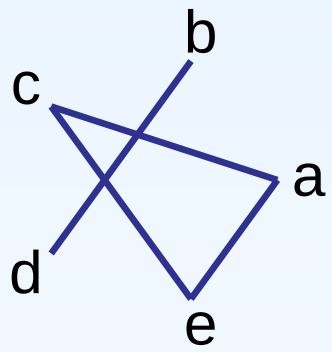

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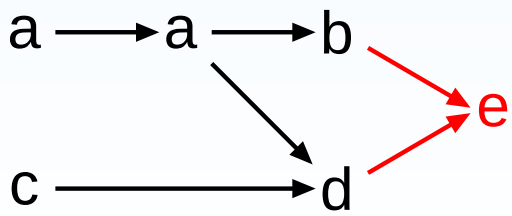

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Unbounded Parallelism

Assume to have a number $\geq |\Sigma|$ of processors. Let $t \in \Sigma^*/\equiv$:

- The length $l(t)$ is the length of w, where $t = [w]$.
- The height $h(t)$ is the length of its Foata Normal Form.

Look at $t = [acabde] = (\{a, c\}, \{a\}, \{b, d\}, \{e\})$ as a process:

$l(t) =$ Sequential time

$h(t) =$ Parallel time

$$\text{speedup of } t = \frac{l(t)}{h(t)} = \frac{\text{sequential time}}{\text{parallel time}}$$

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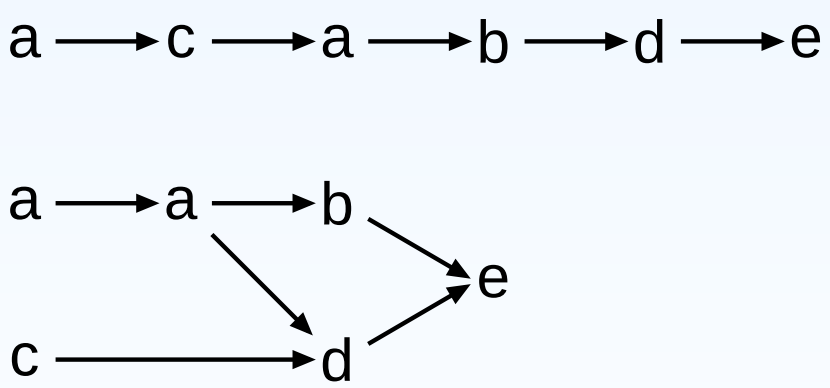

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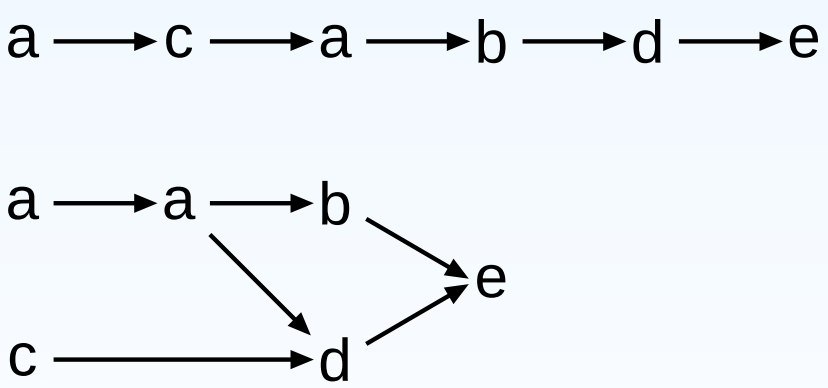

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Mean (Average) Speedup

Mean Speedup over the words of equal length

Uniform Distribution over Σ^n:

$$\lambda_*(\Sigma, C) = \lim_{n\to\infty} \frac{1}{|\Sigma|^n} \sum_{w\in\Sigma^n} \frac{n}{h([w])}$$

Same process execution time (sequential framework).

Mean Speedup over the traces of equal height

Uniform Distribution over $h^{-1}(n)$:

$$\eta_{\mathcal{M}}(\Sigma, C) = \lim_{n\to\infty} \frac{1}{|h^{-1}(n)|} \sum_{t\in h^{-1}(n)} \frac{l(t)}{n}$$

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The MSWL and MSTH Problems

Name: MSWL (Mean Speedup on Words of equal Length)
Input: a parallel alphabet (Σ, C).
Output: $\lambda_*(\Sigma, C)$.

- $\lambda_*(\Sigma, C)$ exists [Sayeb, 89]; and is not algebraic in general [Krob et al., 03];
- MSWL is a restriction of a NP-hard approximable problem (max-algebra spectral radius problem) [Blondel et al., 00];

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Approximability and non-approximability results

MSWL admits a *probabilistic* fully polynomial time approximation scheme (pFPTAS), i.e.

Theorem ([B./R. 2007])

There exists a randomized algorithm A, taking as input (Σ, C) and an error $0 < \epsilon < 1$, such that

- *A is polynomial in $|\Sigma|$ and $1/\epsilon$.*
- *with probability at least $1/2$,*

$$(1+\epsilon)^{-1} A(\Sigma, C, \epsilon) \leq \lambda_*(\Sigma, C) \leq (1+\epsilon) A(\Sigma, C, \epsilon).$$

MSTH is not efficiently approximable:

Theorem ([B./R. 2007])

Unless NP $=$ coR, MSTH problem is hard to approximate within a factor $|\Sigma|^{1-\epsilon}$.

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Input: a graph (Σ, C), an integer in unary repr. 1^k.
Output: $t \in \mathcal{M}_{(\Sigma,C)}$, $h(t) = k$, with probability $1/\left|h^{-1}(k)\right|$.

Our Contribution:

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Let A be a polynomial time random generator for URGT.

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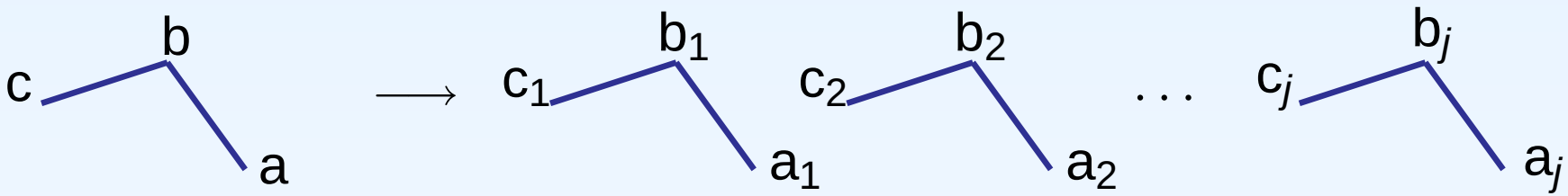

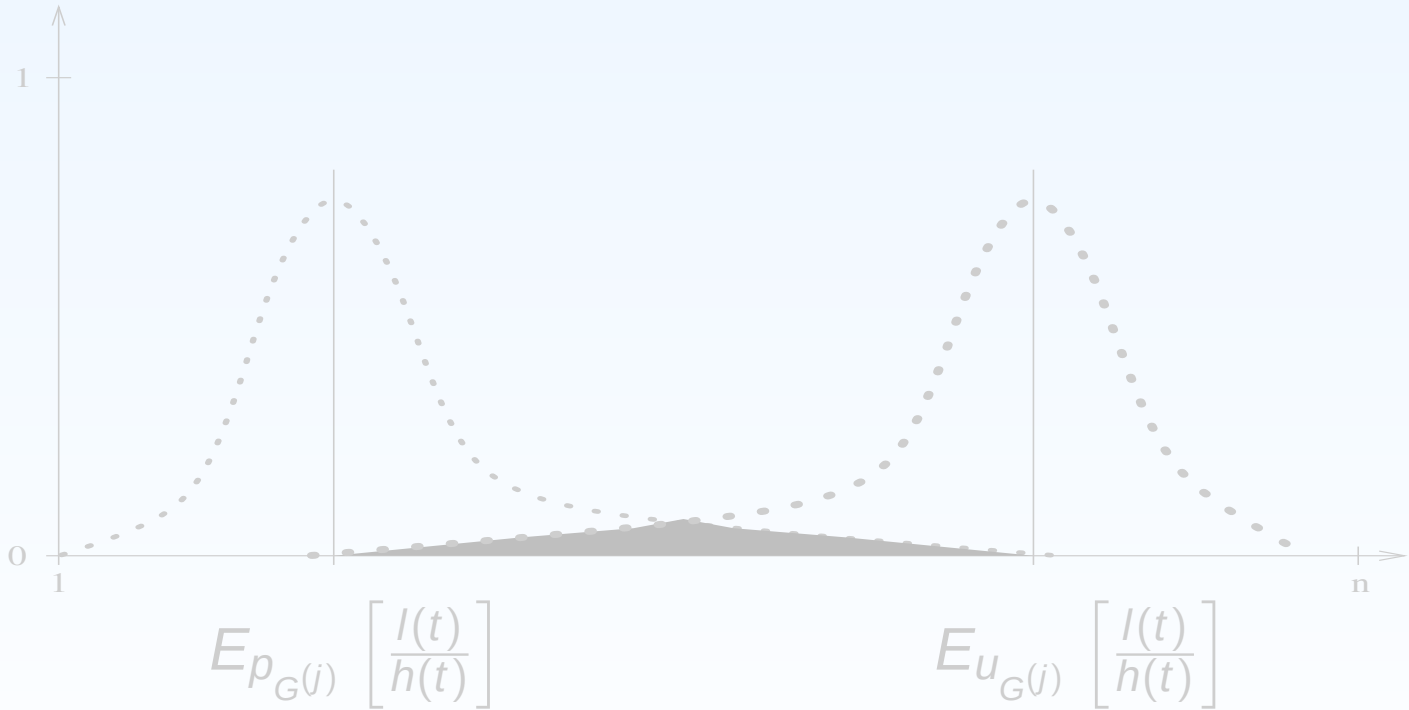

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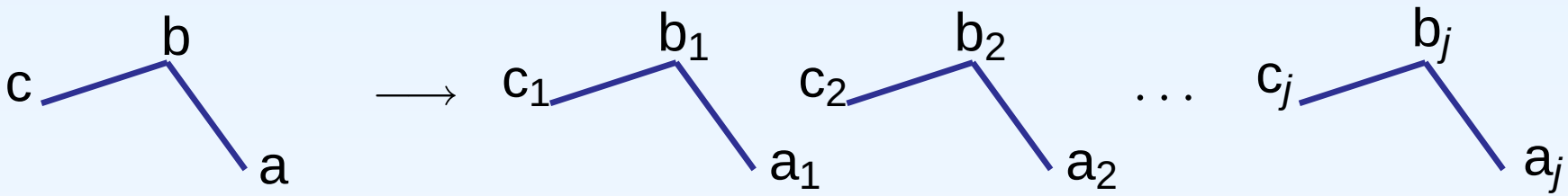

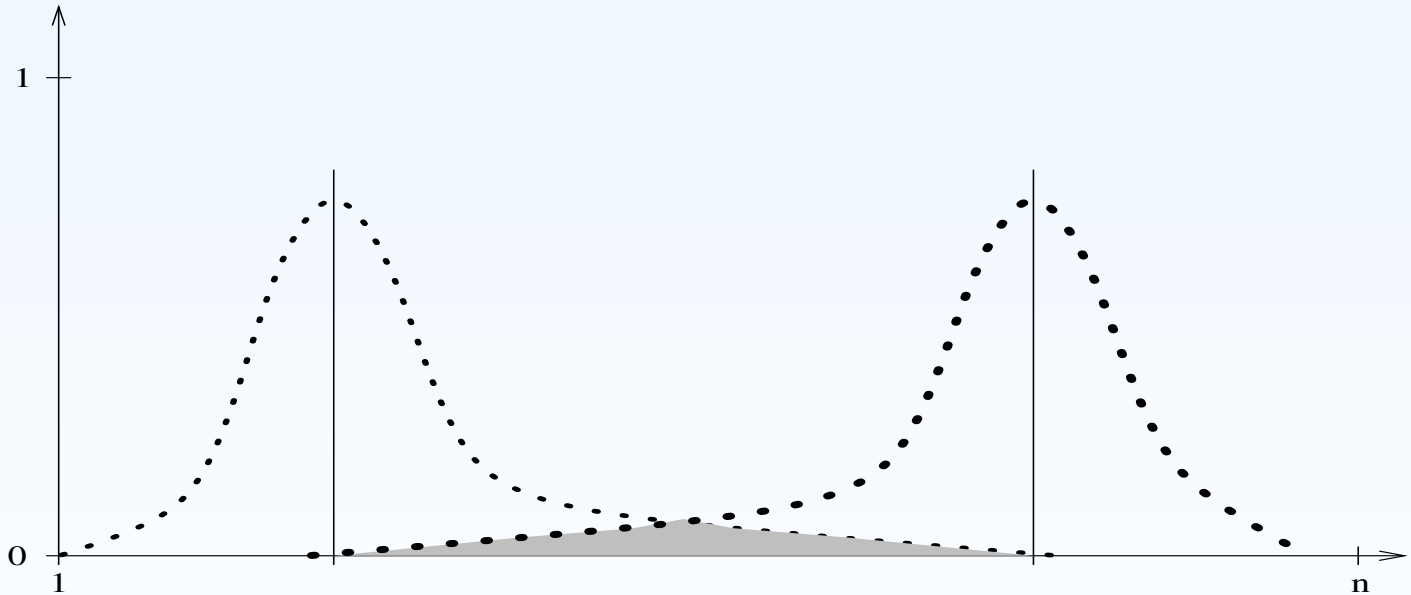

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Series-Parallel Graphs

Question: is there a subclass of graphs for which URGT admits a polynomial solution?

Answer: Yes. Series-parallel graphs.

Consider $G_1 = (\Sigma_1, C_1)$ and $G_2 = (\Sigma_2, C_2)$ with $\Sigma_1 \cap \Sigma_2 = \emptyset$:

- *Series composition* $G_1 \cdot G_2 = (\Sigma_1 \sqcup \Sigma_2, C_1 \sqcup C_2)$.

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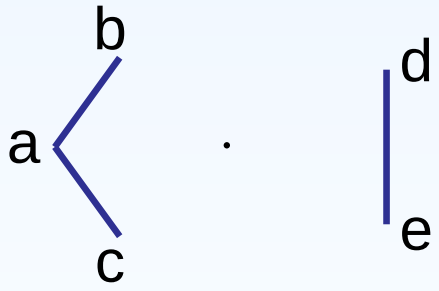

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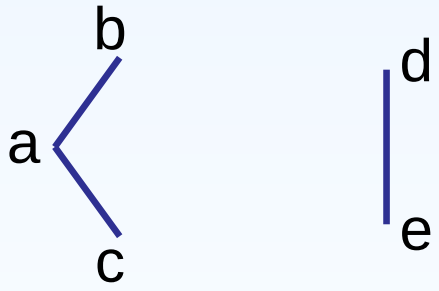

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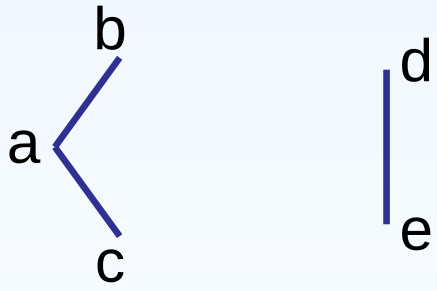

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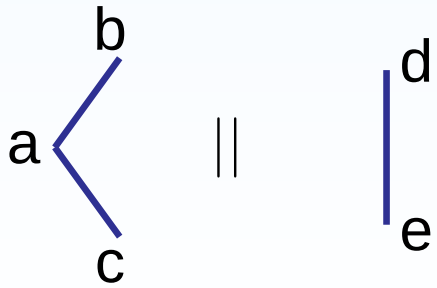

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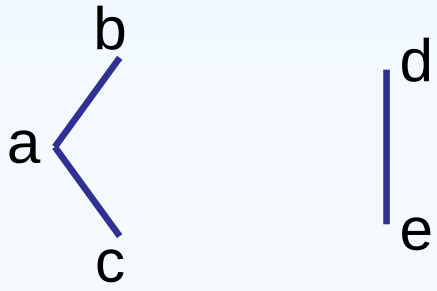

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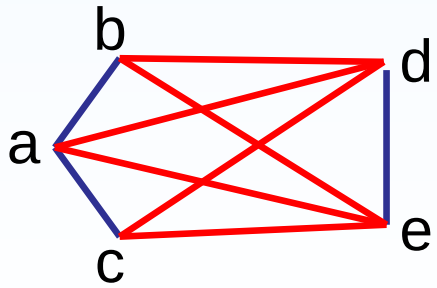

The SPG class

Series-Parallel Graphs (SPG) can be defined recursively:

- $(\{a_1\}, \{\}) \in$ SPG, $\forall i \in \mathbf{N}$;
- $(\Sigma_1, C_1)||(\Sigma_2, C_2)$, $(\Sigma_1, C_1) \cdot (\Sigma_2, C_2) \in$ SGP if $\Sigma_1 \cap \Sigma_2 = \emptyset$.

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Uniform random generation of $t \in \mathcal{M}_G$ of height k:

$G = G_1 || G_2$

If $G = (\{a_i\}, \{\})$, then $t = a_i^k$.

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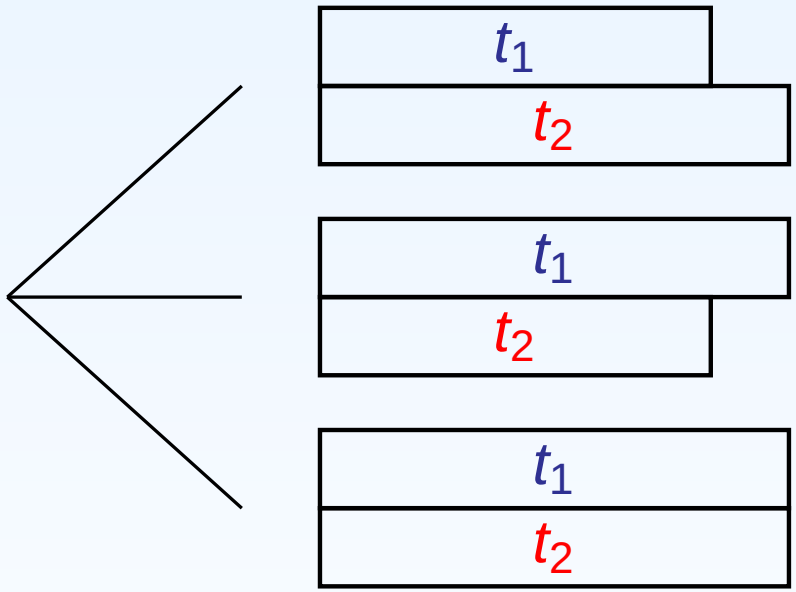

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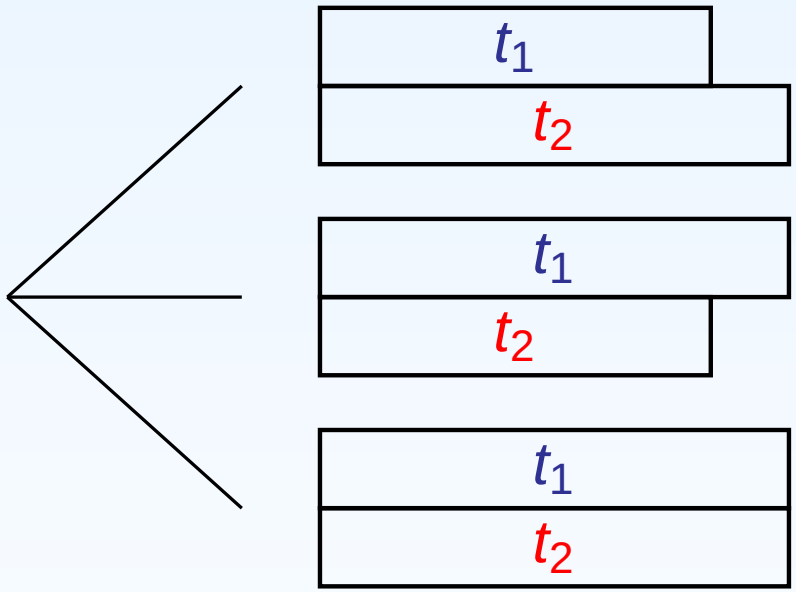

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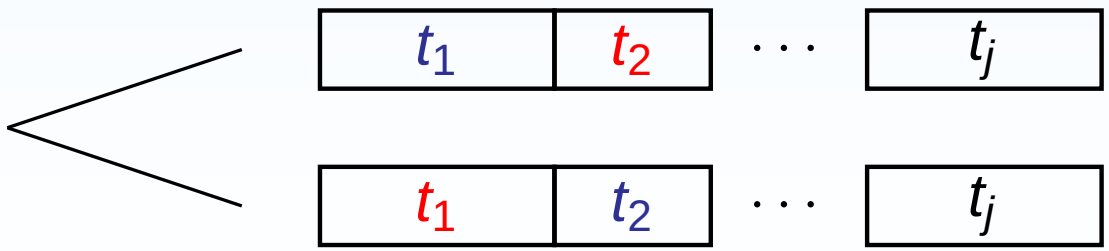

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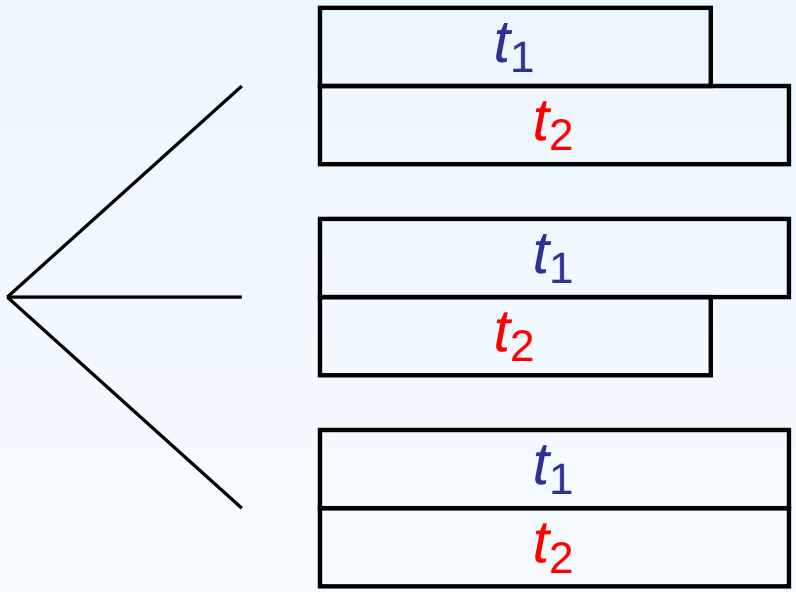

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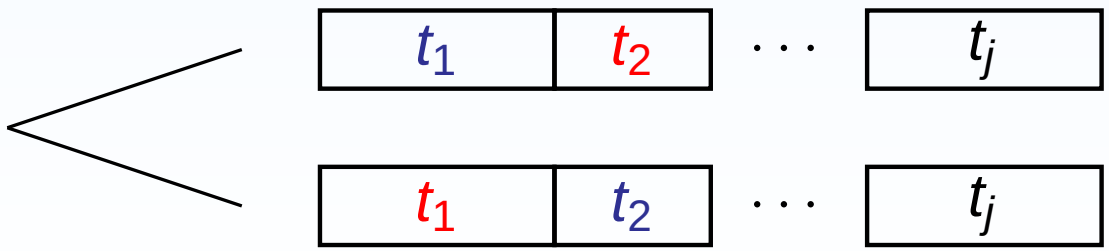

Thank you